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Monotonicity Results on the Zeros of Generalized Laguerre Polynomials*

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It is shown that, for $\alpha > -1$ and $n = 1, 2, \ldots$, the sequence $(n + (\alpha + 1)/2)\, x_{nk} - \frac{1}{4} x_{nk}^2$ increases with n, where $x_{nk} = x_{nk}^{(\alpha)}$ denotes the kth zero of the generalized Laguerre polynomial, in increasing order. As a consequence of this result, the inequality $x_{nk} x_{n+m,k+1} < x_{n,k+1} x_{n+m,k}$, $m = 1, 2, \ldots$, $1 \leqslant k < k + 1 \leqslant n$, is established. Similar results are proved for the zeros of Hermite polynomials. The principal tool used is Sturm's comparison theorem in a variation due to Szegö.

1. INTRODUCTION

For $\alpha > -1$ and $n = 1, 2, \ldots$, let $x_{nk} = x_{nk}^{(\alpha)}$ denote the kth zero of the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$, in increasing order.

A well-known result asserts that x_{nk} is positive and decreases with increasing n. Moreover, Szegö established [6, p. 129] the stronger property that $(n - (\alpha + 1)/2)\, x_{nk}$ decreases with increasing n, i.e.,

$$\left(n + \frac{\alpha + 1}{2}\right) x_{nk} > \left(n + 1 + \frac{\alpha + 1}{2}\right) x_{n+1,k}, \qquad \alpha > -1,\ k = 1, 2, \ldots, n. \tag{1.1}$$

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Similar investigations were made in [1, p. 251] where, among other things, it was shown that, for fixed k, $(n+(\alpha+1)/2)\ (x_{n,k+1}-x_{nk})$ decreases with increasing n. Clearly this result recovers Lorch's result [4] that $x_{n,k+1}-x_{nk}$ decreases with n provided $-1<\alpha\leqslant 1$.

It is worth mentioning that the proofs of every result recalled above is based on the Sturm comparison theorem in the following Szegö's form [6, p. 19].

LEMMA. *Let the functions $y(x)$ and $Y(x)$ be nontrivial solutions of the differential equations*

$$y''+f(x)\,y=0, \qquad Y''+F(x)\,Y=0$$

and let them have consecutive zeros at $x_1, x_2,..., x_m$ and $X_1, X_2,..., X_m$ respectively on an interval (a, b). Suppose that f and F are continuous, that

$$f(x)<F(x), \qquad a<x<x_m$$

and that

$$\lim_{x\to a^+}\ [y'(x)\,Y(x)-y(x)\,Y'(x)]=0. \tag{1.2}$$

Then

$$X_k<x_k, \qquad k=1, 2,..., m.$$

This Lemma will be useful also in the proofs of Theorems 1.1–1.2 of the present paper where we are interested in the monotonicity results in n of quantity $(n+(\alpha+1)/2)\,x_{nk}-\frac{1}{4}x_{nk}^2$, $n=1, 2,...$ and in the determinantal inequality

$$\begin{vmatrix} x_{nk} & x_{n,k+1} \\ x_{n+1,k} & x_{n+1,k+1} \end{vmatrix}<0.$$

Precisely we shall prove the following results:

THEOREM 1.1. *For $-1<\alpha\leqslant 1$ and $k=1, 2,..., n$, let $x_{nk}=x_{nk}^{(\alpha)}$ be the kth positive zero of the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ in increasing order. Then $(n+(\alpha+1)/2)\,x_{nk}-\frac{1}{4}x_{nk}^2$ increases with n, i.e.,*

$$\left(n+1+\frac{\alpha+1}{2}\right)x_{n+1,k}-\frac{1}{4}\,x_{n+1,k}^2>\left(n+\frac{\alpha+1}{2}\right)x_{nk}-\frac{1}{4}\,x_{nk}^2. \tag{1.3}$$

THEOREM 1.2. *For a fixed* α, $-1<\alpha\leqslant 1$ *and* $k=1,2,\ldots,n$, *let* $x_{nk}=x_{nk}^{(\alpha)}$ *be defined as above. Then the following determinantal inequality*

$$\begin{vmatrix} x_{nk} & x_{n,k+1} \\ x_{n+m,k} & x_{n+m,k+1} \end{vmatrix} < 0, \qquad 1\leqslant k<k+1\leqslant n, \quad m=1,2,\ldots \tag{1.4}$$

holds.

We observe that results similar to (1.4) have been proved for the positive zeros of ultrapherical polynomials [2], zeros of Bessel functions [5] and the derivative of Bessel functions [3].

2. Proof of the Theorems

The function $y=y_n(x)=e^{-x/2}x^{(\alpha+1)/2}L_n^{(\alpha)}(x)$ satisfies the differential equation [6, p. 100]

$$y''+f_n(x)\,y=0, \tag{2.1}$$

where

$$f_n(x)=\frac{n+\dfrac{\alpha+1}{2}}{x}+\frac{1-\alpha^2}{4x^2}-\frac{1}{4}. \tag{2.2}$$

Let us carry out the following transformation

$$y(x)=a(x)\,z(t) \tag{2.3}$$

$$t=\int_{x'}^{x} b(x)\,dx \tag{2.4}$$

where $a(x)$, $b(x)$ are positive, continuous, twice differentiable functions on $I=[x',x'']$. If $a^2(x)\,b(x)\equiv 1$ on I, then $z=z(t)$ is a solution of the differential equation

$$z''+F_n(t)z=0 \tag{2.5}$$

where

$$F_n(t)=\frac{3b'^2-2bb''}{4b^4}+\frac{f_n(x)}{b^2}. \tag{2.6}$$

In the proofs of our theorems we shall use two transformations of (2.1).

Proof of Theorem 1.1. Let $p_n(x)$ be defined by

$$p_n(x) = \left(n + \frac{\alpha+1}{2}\right) x - \frac{1}{4} x^2. \tag{2.7}$$

It is clear that

$$\max_{-\infty < x < \infty} p_n(x) = p_n(\tilde{x}_n) = \left(n + \frac{\alpha+1}{2}\right)^2$$

where

$$\tilde{x}_n = 2n + \alpha + 1.$$

Moreover

$$p_{n+1}(x) > p_n(\tilde{x}_n) \qquad \text{for} \quad \tilde{x}_{n1} < x < \tilde{x}_{n2} \tag{2.8}$$

where

$$\tilde{x}_{n1} = \tilde{x}_n + 2 - \sqrt{4\tilde{x}_n + 4}, \qquad \tilde{x}_{n2} = \tilde{x}_n + 2 + \sqrt{4\tilde{x}_n + 4}.$$

Now we distinguish three cases:

(a) $\tilde{x}_{n1} < x_{n+1,k} < \tilde{x}_{n,2}$

(b) $\tilde{x}_{n2} \leqslant x_{n+1,k}$

(c) $0 < x_{n+1,k} \leqslant \tilde{x}_{n1}$.

On using (2.7), inequality (1.3) can be written as

$$p_n(x_{nk}) < p_{n+1}(x_{n+1,k}), \qquad k = 1, 2,..., n. \tag{2.9}$$

Case a. By (2.8) we have immediately inequality (2.9).

Case b. By (1.1) we get

$$x_{n+1,k} < x_{nk} \frac{2n+\alpha+1}{2n+\alpha+3},$$

moreover by (2.7) the function $p_{n+1}(x)$ decreases with respect to x on $[\tilde{x}_n + 2, \infty)$. Therefore in order to prove (2.9) it is sufficient to show that $p_n(x_{nk}) < p_{n+1}(x_{nk}(2n+\alpha+1)/(2n+\alpha+3))$, which can be verified by direct computation.

Case c. In (2.3) we choose $a(x)$ and $b(x)$ as

$$b(x) = b_n(x) = n + \tfrac{1}{2}(\alpha+1) - \tfrac{1}{2}x; \quad a(x) = a_n(x) = [b(x)]^{-1/2}.$$

Letting $x'=0$ in (2.4) we find that the function $z_n(t)=\sqrt{b_n(x)}\ y_n(x)$ satisfies the differential equation

$$z''(t)+F_n(t)\,z(t)=0, \tag{2.10}$$

where

$$F_n(t)=\frac{4t+1-\alpha^2}{4x^2b_n^2(x)}+\frac{3}{16}\frac{1}{b_n^4(x)} \tag{2.11}$$

and

$$x=x_n(t)=2\{n+\tfrac{1}{2}(\alpha+1)-\sqrt{[n+\tfrac{1}{2}(\alpha+1)]^2-t}\}.$$

Besides (2.10) we consider the differential equation

$$w''(t)+F_{n+1}(t)\,w(t)=0 \tag{2.12}$$

satisfied by $z_{n+1}(t)$. If we put $x''=\tilde{x}_{n1}$ in (2.4) then $b_n(x)$ and $b_{n+1}(x)$ are positive on $I=[0,\tilde{x}_{n1}]$. Now we claim that (2.10) is a Sturmian majorant of (2.12). This will be verified if we show that the functions $x_n(t)\,b_n(x)$ and $b_n(x)$ are increasing with respect to n. We get

$$\frac{\partial x_n(t)}{\partial n}=\frac{-2x}{2n+\alpha+1-x}<0$$

and

$$\frac{\partial b_n(x)}{\partial n}=\frac{2n+\alpha+1}{2n+\alpha+1-x}>0.$$

Moreover

$$xb_n(x)=[n+\tfrac{1}{2}(\alpha+1)]x-\tfrac{1}{2}x^2-t-\tfrac{1}{4}x^2$$

hence $b_n(x)$ and $xb_n(x)$ are both increasing with n, when t is fixed.

The limit condition (1.2) in Lemma is satisfied at $t=0$. Then an application of Sturm comparison theorem gives that between the zeros of $z_n(t)$ and $z_{n+1}(t)$ the inequalities

$$t_{nk}<t_{n+1,k}, \qquad k=1,2,\ldots,n$$

hold, where $t_{nk}=p_n(x_{nk})$, $t_{n+1,k}=p_{n+1}(x_{n+1,k})$.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Applying the transformation $t=x/x_{nk}$ in the differential equation (2.1) (i.e., $b(x)=1/x_{nk}$ in (2.4)) we find by (2.6) that the function $z_n(t)=y_n(x)$ satisfies

$$z''+x_{nk}^2f_n(x_{nk}t)z=0 \tag{2.13}$$

hence by (2.2)

$$x_{nk}^2 f_n(x_{nk} t) = \frac{n + \frac{1}{2}(\alpha + 1)}{t} x_{nk} + \frac{1 - \alpha^2}{4t^2} - \frac{1}{4} x_{nk}^2 \tag{2.14}$$

Besides (2.13) we consider the equation

$$w'' + x_{n+1,k}^2 f_{n+1}(x_{n+1,k} t) w = 0 \tag{2.15}$$

satisfied by $z_{n+1}(t) = y_{n+1}(x)$. The functions $z_n(t)$ and $z_{n+1}(t)$ have a common zero at $t = 1$. Moreover for $t > 1$ we have

$$\begin{aligned} t[x_{n+1,k}^2 f_{n+1}(x_{n+1,k} t) - x_{nk}^2 f_n(x_{nk} t)] \\ > [n + 1 + \tfrac{1}{2}(\alpha + 1)]\, x_{n+1,k} - \tfrac{1}{4} x_{n+1,k}^2 \\ - [n + \tfrac{1}{2}(\alpha + 1)]\, x_{nk} + \tfrac{1}{4} x_{nk}^2 . \end{aligned}$$

and the right-hand side is positive in view of Theorem 1.1. Therefore (2.15) is a Sturmian majorant of (2.13) and an application of Lemma gives that the next zeros of $z_{n+1}(t)$ occur before the next zeros of $z_n(t)$, that is

$$\frac{x_{n+1,k+l}}{x_{n+1,k}} < \frac{x_{n,k+l}}{x_{nk}}, \qquad l = 1, 2,..., n - k. \tag{2.16}$$

This proves (1.4) for $m = 1$. Then step by step we obtain from (2.16) the more general inequality

$$\frac{x_{n+m,k+l}}{x_{n+m,k}} < \frac{x_{n,k+l}}{x_{nk}}, \qquad m = 1, 2,... . \tag{2.17}$$

which completes the proof of Theorem 1.2.

Now we apply our results to the zeros of Hermite polynomials. Let $\xi_{n,k}$, $k = 1, 2,..., [n/2]$ be the positive zeros of the Hermite polynomial $H_n(x)$ in increasing order, i.e., $0 < \xi_{n1} < \xi_{n2} < \cdots < \xi_n$, $[n/2]$. An immediate consequence of Theorem 1.2 is the following.

Corollary. *Let* ξ_{nk} $(k = 1, 2,..., [n/2])$ *be the positive zeros of the Hermite polynomial* $H_n(x)$ *in increasing order. Then the inequality*

$$\begin{vmatrix} \xi_{n,k} & \xi_{n,k+l} \\ \xi_{n+2m,k} & \xi_{n+2m,k+l} \end{vmatrix} < 0$$

holds, for $k = 1, 2,..., [n/2]$ *and* $m = 1, 2,...$.

Proof. By the relations between Hermite polynomials and Laguerre polynomials [6, p. 106]

$$H_{2n}(x) = (-1)^n 2^n n! L_n^{(-1/2)}(x^2)$$
$$H_{2n+1}(x) = (-1)^n 2^{n+1} n! x L_n^{(1/2)}(x^2)$$

we have

$$\xi_{2n,k} = \sqrt{x_{nk}^{(-1/2)}}, \qquad \xi_{2n+1,k} = \sqrt{x_{nk}^{(1/2)}}, \quad k = 1, 2, ..., n.$$

Then (2.17) implies our Corollary.

References

1. S. AHMED, A. LAFORGIA, AND M. E. MULDOON, On the spacing of the zeros of some classical orthogonal polynomials, *J. London Math. Soc.* (2), **25** (1982), 246–252.
2. Á. ELBERT AND A. LAFORGIA, Some monotonicity properties of the zeros of ultraspherical polynomials, *Acta Math. Hungar.* **48** (1986), 155–159.
3. A. LAFORGIA, Sturm theory for certain classes of Sturm-Liouville equations and Turánians and Wronskians for the zeros of derivative of Bessel functions, *Indag. Math.* **44** (1982), 295–301.
4. L. LORCH, Elementary comparison techniques for certain classes of Sturm. Liouville equation, *in* "Proc. Uppsala 1977, Inter. Conf. Diff. Equations," Symposia Univ. Upsaliensis Annum Quingentesimum celebrantis Vol. 7, pp. 125–133. Acta Univ. Upsaliensis, Uppsala 1977.
5. L. LORCH, Turánians and Wronskians for the zeros of Bessel functions, *SIAM J. Math. Anal.* **11** (1980), 223–227.
6. G. SZEGÖ, Orthogonal Polynomials, 4th ed., *in* "Amer. Math. Soc., Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, R. I., 1975.